

Dynamic optimal capital growth of diversified investment

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We investigate the problem of dynamic optimal capital growth of diversified investment. A general framework that the trader maximize the expected log utility of long-term growth rate of initial wealth was developed. We show that the trader's fortune will exceed any fixed bound when the fraction is chosen less than critical value. But, if the fraction is larger than that value, ruin is almost sure. In order to maximize wealth, we should choose the optimal fraction at each trade. Empirical results with real financial data show the feasible allocation. The larger the fraction and hence the larger the chance of falling below the desired wealth growth path.

Keywords: capital growth; asset allocation; portfolio optimization; diversification; reinvestment

1. Introduction

The use of log utility dates at least to the article of Daniel Bernoulli in 1738 [1]. The idea is to determine the expected value and what you would pay for the following game: A fair coin with 0.5 probability of heads is repeatedly tossed until heads occurs, ending the game. The gambler receives in return 2^{k-1} with probability 2^{-k} for $k = 1, 2, \dots$ if a head occur. After each loss, the bet is doubled to 2, 4, 8, ... etc. Clearly, the expected value is $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots$ or infinity with linear utility.

St Petersburg game is attractive but the gambler wants less as bet approach infinity. Bernoulli offers solution since he feels that this gamble is worth a lot less than infinity. When utility is log, as Bernoulli proposed, the expected value is $\frac{1}{2} \ln 1 + \frac{1}{4} \ln 2 + \frac{1}{8} \ln 4 + \dots = \ln 2 = 0.6931$.

Kelly's 1956 technical journal paper presented the criterion about optimal fraction of the gambler's stake which maximizing the growth of the gambler's wealth [6]. The analysis use log utility

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to deal with repeated gambling problem. This is called Kelly betting. For a long-term gambler who make many short-term decisions, the criterion yields the highest long run levels of wealth.

Latane introduced log utility as an investment criterion to the finance world independent of Kelly's work. Focusing on simple intuitive versions of the expected log criteria, he suggested that it had superior long run properties [7]. Breiman established the basic mathematical properties of the expected log criterion. Breiman [2] proved that optimal strategy based on log utility will almost surely beat any different strategy in the long run. Finkelstein and Whitley [5] generalize the Breiman results to independent identically distributed assets that are not necessarily discretely distributed as Breiman assumed.

Thorp discusses the general theory of optimal betting over time on favorable games [12]. Favorable games are those with a strategy such that $Prob[\lim_{n \rightarrow \infty} W_n > W_0] = 1$, where W_n is the gambler's capital after n trials. Thorp follows the footsteps of Kelly and Breiman, by discussing some favorable games such as blackjack, roulette [13].

Cover presents a universal portfolio that will perform as well as if the trader knew the realized distribution of the future asset returns [3]. There are no assumption on the asset returns. The universal portfolio strategy is based on the past returns and will perform asymptotically as well as the portfolio based on foreknowledge of the sequence of price. In recent decades, a good recent overviews of the field can be found in [8–11, 14, 15].

Despite explosive development over multi decades and extensive application to the construction of equity portfolios, modern portfolio theory has found little use in the larger portfolio. There are several reasons for this. First, traditional portfolio theory, being principally based on the mean variance model, is fundamentally single period in nature, while the larger problem focus on multi-period. Second, traditional model rely heavily on assumption of return distribution, real world stochastic processes. At the other extreme, continuous time portfolio theory is somewhat intractable in a world of nontrivial transaction costs, risk and other constraints.

We investigate the problem of dynamic optimal capital growth of diversified investment and extend the basic criterion to multi assets case. The theory and practical application of the criterion is straightforward when the underlying probability distributions are fairly accurately known. However, realized future equity returns may be very different from what one would expect using estimates based on historical returns. Prospective users of the criterion can check the properties to test whether it is well suited to their intended application. Properties of the capital growth investment criterion and the feasible allocation were investigated in this paper.

2. Optimization of capital growth entropy

We assume that the trader has found a positive expectation return, he is able to trade repeatedly, the initial wealth be W_0 , after n turns the wealth is W_n . The winning probability is p , the probability of losing is $q = 1 - p$. We define the return be $r_i = (W_i - W_{i-1})/W_{i-1}$, where W_i is the wealth after i turns.

The amount of money he could make depend only on how much he chose to allocate. How much would he allocate? Furthermore, suppose the trader allocates fraction f_i of the actual wealth in i turn. After n turns the trader's wealth is equal to

$$W_n = W_0 \prod_{i=1}^n (1 + f_i r_i). \quad (1)$$

We use natural logarithms

$$G = \lim_{n \rightarrow \infty} \frac{1}{n} \ln \frac{W_n}{W_0} \quad (2)$$

as a criterion for investment optimization. Due to the multiplicative character of W_n , G can be rearranged as

$$\begin{aligned} G &= \lim_{n \rightarrow \infty} \frac{1}{n} \ln \prod_{i=1}^n (1 + f_i r_i) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \ln(1 + f_i r_i) \\ &= \langle \ln(1 + f_i r_i) \rangle. \end{aligned} \quad (3)$$

Imagine that a trader is faced with an infinitely wealthy opponent who will play repeated independent trials of a biased coin. In this game, it seems that an optimal strategy will involve always allocating the same fraction of your total wealth. To make this possible, we shall assume that capital is infinitely divisible. If we allocate according to $B_i = f_i W_i$, where $0 \leq f_i \leq 1$, this is called ‘fixed fraction’ strategy. Suppose S and F are the number of successes and failures, then our capital after n trials is

$$W_n = W_0(1 + f)^S(1 - f)^F, \quad (4)$$

where $S + F = n$. We refer to a trade’s reward to risk ratio as a R multiple. Based on past experience, the trader knows the likely reward and permissible risk on a trade before its initiation, he can estimate the probability of success. We note that the exponential rate of increase per trial

$$\ln \left[\frac{W_n}{W_0} \right]^{1/n} = \frac{S}{n} \ln(1 + af) + \frac{F}{n} \ln(1 - bf), \quad (5)$$

where a and b is R multiple. For the risky game introduced above, the expected value of Equation (5) can be rearranged as

$$G(f) = p \ln(1 + af) + (1 - p) \ln(1 - bf). \quad (6)$$

Note that

$$G'(f) = \frac{pa}{1 + af} - \frac{(1 - p)b}{1 - bf} = 0. \quad (7)$$

When

$$f = f^* = \frac{pa + pb - b}{ab}. \quad (8)$$

Now

$$G''(f) = -\frac{pa^2}{(1 + af)^2} - \frac{(1 - p)b^2}{(1 - bf)^2} < 0, \quad (9)$$

$G(f)$ has a unique maximum at $f = f^*$, where

$$G(f^*) = p \ln(pa + pb) + (1 - p) \ln \frac{(a + b)(1 - p)}{a} \quad (10)$$

Moreover, $G(f_c) = 0$, so we get unique number $f_c > 0$, where $0 < f^* < f_c < 1$. When $a = b = 1$, the fraction

$$f = f^* = 2p - 1. \quad (11)$$

If $p < \frac{1}{2}$, $f^* < 0$, a short position is suggested. In this paper, we exclude short selling and thus the optimal choice is $f^* = 0$. For $\frac{1}{2} < p < 1$, the maximum of G can be rewritten as

$$G(f^*) = \ln 2 + p \ln p + (1 - p) \ln(1 - p), \quad (12)$$

where

$$S(p) = p \ln p + (1 - p) \ln(1 - p) \quad (13)$$

is the Shannon entropy assigned to the risky game with the winning probability p . The trader introduced here follows a different criterion from the classical trader. Because of the logarithm which is additive in repeated allocates and to which the law of large numbers applies. At every allocate he maximizes the expected value of the logarithm of his capital. Suppose the trader was allowed to allocate one dollar each week but not to reinvest his winnings. He would then maximize his expected value of capital on each turn. He would allocate all his available capital on the event yielding the highest expectation.

Assume a trader with an initial capital of W_0 and capital is infinitely divisible. He wins money on successive independent trading with a win probability of 0.51. Applying the results, $f^* = 2p_i - 1 = 1.02 - 1 = 0.02$. Thus, 2% of current capital should be allocated on each trading in order to cause the initial capital to grow at the fastest rate. If the trader continually allocates a fraction smaller than 2%, W_0 will also grow to infinity but the capital growth will be slower.

A criticism applied to the strategy is that capital is not infinitely divisible but multiples of a minimum unit. If the minimum allocation allowed is small relative to the trader's initial capital, the probability of ruin in the standard sense is negligible. In the security markets, the minimum unit can be as small as desired. Due to this principle, a smart trader will quit the market if he has no enough money.

3. Dynamic optimal capital growth portfolio

3.1 Two asset case

Suppose we play simultaneously on two independent favorable investments with allocating fraction f_1 and f_2 and with success probabilities p_1 and p_2 , failure probabilities q_1 and q_2 . The expected growth rate is given by

$$\begin{aligned} G(f_1, f_2) = & p_1 p_2 \ln(1 + f_1 + f_2) + p_1 q_2 \ln(1 + f_1 - f_2) \\ & + q_1 p_2 \ln(1 - f_1 + f_2) + q_1 q_2 \ln(1 - f_1 - f_2). \end{aligned} \quad (14)$$

To find the optimal f_1^* and f_2^* , we solve the simultaneous equations $\partial G / \partial f_1 = 0$ and $\partial G / \partial f_2 = 0$.

Simultaneous allocates on different markets may be independent but at the same market they have a correlation. This should substantially reduce the fraction per trading. On the other hand, correlations between the returns on securities can range from nearly -1 to nearly 1 . An extreme correlation often can be exploited to great advantage through the techniques of 'hedging'. The risk averse trader may be able to acquire combinations of securities where the expectations add and the risks tend to cancel.

It is important to note that for an exact solution or accurate numerical approximation, we need to use the entire joint distribution to construct the growth function. To illustrate the fraction of the securities, we return to the study of the effects of correlation. Consider the pair of assets X_1 and X_2 , with joint distribution given in Table 1. The growth function

$$G(f_1, f_2) = a \ln[1 + 2(m + 1)f] + (1 - 2a) \ln(1 + 2mf) + a \ln[1 + 2(m - 1)f]. \quad (15)$$

Clearly, $0 \leq a \leq \frac{1}{2}$ and $\text{Cov}(X_1, X_2) = \text{Cor}(X_1, X_2) = 4a - 1$ increases from -1 to 1 as a increases from 0 to $\frac{1}{2}$. Finding a general solution for f_1^*, f_2^* appears algebraically complicated, but specific solutions are very easy to find numerically.

Table 1. Joint distribution of X_1 and X_2 .

	X_2	
	$m_2 + 1$	$m_2 - 1$
X_1	$m_1 + 1$	$\frac{1}{2} - a$
	$m_1 - 1$	$\frac{1}{2} + a$

Consider the instance when $a = 0$ so $\text{Cor}(X_1, X_2) = -1$. Then, $G(f) = \ln(1 + 2mf)$ which increases without limit as f increases and one should allocate as much as possible. This is a simplified version of the classic arbitrage of securities markets: find a pair of securities which are identical and trade at disparate prices. Buy the relatively underpriced security and sell short the relatively overpriced security, achieving a correlation of -1 and locking in a riskless profit.

In applying the criterion to the securities markets, we meet new analytic problems. Allocation on a security typically has many outcomes rather than just a few, as in most gambling situations. This lead to the use of continuous instead of discrete probability distributions. Frequently, the problem is to find an optimum portfolio from among n securities, where n may be a large number. We also have constraints and always need $1 + fr > 0$ so $\ln(\cdot)$ is defined, and $\sum f_i = 1$ to limit no borrow. The maximization problem is generally solvable-based growth function is concave.

More about the problem may be found in the following continuous approximation. Let X be a random variable with $P(X = \mu + s) = P(X = \mu - s) = 0.5$. Then, $E(X) = \mu$, $\text{Var}(X) = s^2$. With initial capital W_0 , allocating fraction f , and return per unit is r , the wealth after first trial is

$$W_1 = W_0(1 + (1 - f)r_0 + fr) = W_0(1 + r_0 + f(r - r_0)), \quad (16)$$

where r_0 is the rate of return on the remaining capital, invested in riskless asset. Then,

$$\begin{aligned} G(f) &= E\left(\ln \frac{W_1}{W_0}\right) = E[\ln(1 + r_0 + f(r - r_0))] = 0.5 \ln(1 + r_0 + f(r - r_0 + s)) \\ &\quad + 0.5 \ln(1 + r_0 + f(r - r_0 - s)). \end{aligned} \quad (17)$$

Now, subdivide the time interval into n equal independent steps, keeping the same drift and the same total variance. Thus, μ, s^2, r are replaced by $\mu/n, s^2/n, r/n$. We have n independent $X_i, i = 1, 2, \dots, n$ with

$$P\left(X_i = \frac{\mu}{n} + \frac{s}{\sqrt{n}}\right) = P\left(X_i = \frac{\mu}{n} - \frac{s}{\sqrt{n}}\right) = 0.5. \quad (18)$$

Then,

$$V_n = W_1 = W_0 \prod_{i=1}^n (1 + (1 - f)r_0 + fr_i). \quad (19)$$

Taking $E(\ln(\cdot))$ of both sides and expanding the result in a power series leads to

$$G(f) = r_0 + f(\mu - r_0) - \frac{s^2 f^2}{2} + O(n^{-12}). \quad (20)$$

Letting $n \rightarrow \infty$ we have

$$G_\infty(f) = r_0 + f(\mu - r_0) - \frac{s^2 f^2}{2}, \quad (21)$$

$G_\infty(f)$ is the growth rate of capital with fraction f . There is nothing special about the choice of the random variable. Any bounded random variable with mean μ and variance s^2 will lead to

the same result. Note that f no longer needs to be less than or equal to 1. Also, $f < 0$ causes no problems, this simply corresponds to selling the security short. Take derivative, the results are

$$\begin{aligned} f^* &= \frac{\mu - r_0}{s^2}, \\ G_\infty(f^*) &= \frac{(\mu - r_0)^2}{2s^2} + r_0. \end{aligned} \quad (22)$$

Note that when $\mu = r_0 + s^2$ in which case the trader will select the market portfolio without borrowing or lending. If $\mu > r_0 + s^2$ the trader will use leverage and if $\mu < r_0 + s^2$ he will allocate partly in riskless asset and partly in the market portfolio.

When $f = 1$, we have $G_\infty(1) = \mu - s^2/2$ so the portfolio in the (μ, s) plane satisfying $\mu - s^2/2 = C$, where C is a constant, all have the same growth rate. The trader appears to have the utility function $U(\mu, s) = \mu - s^2/2$. Thus, for any bounded set of portfolios, the best portfolios are exactly those in the subset that maximizes the $U(\mu, s)$.

3.2 Many assets with multi-period and reinvestment

Consider that there are $n + 1$ investment opportunities $r_{t0}, r_{t1}, \dots, r_{tm}$ at time t , which can be done simultaneously in each time step, r_{t0} represent riskless asset. In opportunities $i (i = 0, 1, \dots, n)$, the trader allocate the fraction f_{ti} of the total wealth. Let $F_t^* = (f_{t0}^*, f_{t1}^*, \dots, f_{tm}^*)$ be optimal investment strategy. We assume that $f_{t0}^* + f_{t1}^* + \dots + f_{tm}^* \leq 1$ so there is no borrowing. Let the probability of the outcome $r_{t0}, r_{t1}, \dots, r_{tm}$ be $p_{t0}, p_{t1}, \dots, p_{tm}$. The capital growth function

$$G(f_{t0}, f_{t1}, \dots, f_{tm}) = \sum [p_t(\cdot) \ln(1 + f_{t0}r_{t0} + f_{t1}r_{t1} + \dots + f_{tm}r_{tm})], \quad (23)$$

where $p_t(\cdot)$ is joint distribution of n assets with correlated outcomes, $p_t(\cdot) = p_{t0}p_{t1} \dots p_{tm}$ if the returns r_{ti} are independent. Note that concave function $G(f_{t0}, f_{t1}, \dots, f_{tm})$ is defined if and only if $1 + f_{t0}r_{t0} + f_{t1}r_{t1} + \dots + f_{tm}r_{tm} > 0$. Computational procedures for finding an optimal fixed fraction strategy are based on the theory of concave functions. To find the optimal f_{ti}^* , let $\partial G / \partial f_{ti} = 0$. For portfolios with many securities the volume of inputs is prohibitive.

Although, it can be solved efficiently using standard quadratic programming techniques. But many features of the model are subject to a lot of criticism, such as calculating of the expected returns and correlations. Typically, it is measured from historical data and fed into an optimizer as if they are known perfectly, but in fact these data are measured with error. The fat tail and asymmetry of asset returns are in conflict with the normality hypothesis of mean and variance.

For our simultaneous risky investment, the optimal fraction would be the solution of the following optimization problem:

$$\max G(f_{t0}, f_{t1}, \dots, f_{tm}) \quad (24)$$

subject to the following constraints:

$$\begin{aligned} \sum_{i=0}^n f_{ti} &= 1, \\ f_{ti} &\geq 0, \\ 1 &\leq i \leq n, \\ 1 &\leq t \leq T. \end{aligned} \quad (25)$$

The first constraint, called budget constraint, require that all of the budget be invested in the risky portfolio and risk free asset. The nonnegativity constraints express that no short sales are

allowed. Note that the optimal growth function

$$\begin{aligned}
 G(f_{i0}, f_{i1}, \dots, f_{in}) &= \sum [p_i(\cdot) \ln(1 + f_{i0}r_{i0} + f_{i1}r_{i1} + \dots + f_{in}r_{in})] \\
 &= E \left\{ \ln \left[\frac{W(f_{i0}, f_{i1}, \dots, f_{in})}{W_0} \right]^{1/t} \right\} \\
 &= \left(\frac{1}{t} \right) E[\ln W(f_{i0}, f_{i1}, \dots, f_{in})] - \left(\frac{1}{t} \right) \log W_0.
 \end{aligned} \tag{26}$$

The wealth of portfolio at time t

$$W(f_{i0}, f_{i1}, \dots, f_{in}) = W_0 \prod_{i=1}^T \sum_{i=1}^n (1 + f_i r_i). \tag{27}$$

So for a fixed period t , maximizing $G(f_{i0}, f_{i1}, \dots, f_{in})$ is the same as maximizing $(1/t)E[\log W(f_{i0}, f_{i1}, \dots, f_{in})]$. In such situations, classical quadratic programming do not work efficiently and heuristic optimization techniques may be the better choice.

4. Analysis of the optimal capital growth portfolio

4.1 Properties of the criterion

The two most familiar utility function in portfolio optimization is power utility and mean variance utility. Consider the narrow power utility function

$$\begin{aligned}
 U(W) &= \frac{b}{b-1} W^{1-1/b}, \quad W > 0, b > 0, \\
 U'(W) &= W^{-1/b}, \quad U''(W) = -\frac{1}{b} W^{-1/b-1}.
 \end{aligned} \tag{28}$$

Note the Arrow–Pratt risk aversion index

$$\begin{aligned}
 R_A(W) &= -\frac{U''(W)}{U'(W)} = \frac{1}{b} W^{-1}, \\
 R_R(W) &= R_A(W) W = \frac{1}{b}
 \end{aligned} \tag{29}$$

and

$$\begin{aligned}
 \frac{dR_A(W)}{dW} &= -\frac{1}{b} W^{-2} < 0, \\
 \frac{dR_R(W)}{dW} &= 0.
 \end{aligned} \tag{30}$$

The decreasing absolute risk aversion implies that the risky asset is a normal good, the demand for the risky asset increases as the trader's wealth increases. Absolute risk aversion is a measure of the intensity of a trader's aversion to risk, the higher a trader's absolute risk aversion, the higher the minimum risk premium required. The utility has essentially zero risk aversion as wealth approach infinity. Let $(b-1)/b \rightarrow 0$, narrow power utility function converges to log

Table 2. Two-point probability distributions.

	X_1		X_2	
x	-1	$\frac{3}{2}$	-2	$\frac{4}{3}$
p	$\frac{1}{5}$	$\frac{4}{5}$	$\frac{1}{10}$	$\frac{9}{10}$

utility

$$\lim_{(b-1)/b \rightarrow 0} \frac{W^{(b-1)/b} - 1}{\frac{b-1}{b}} = \ln W. \quad (31)$$

So, log utility is a special case of power utility. As $(b-1)/b$ gets larger, the trader is less aggressive since his Arrow–Pratt risk aversion is higher. For a given $(b-1)/b$ and b between 0 and 1, will provide the same portfolio when b is invested in the risky portfolio and $1-b$ is invested in cash.

Consider the negative power utility function $1/\delta W^\delta$ for $\delta < 0$. As δ gets larger negatively, the trader's Arrow–Pratt risk aversion is higher. So you can pick an appropriate δ according to your risk preference. For example, half strategies is $\delta = -1$ and quarter strategies is $\delta = -3$. To pick δ continuously in time so that wealth will stay above a desired wealth growth path.

Log utility also has essentially zero risk aversion, its Arrow–Pratt risk aversion index is

$$R_A(W) = -U''(W)/U'(W) = \frac{1}{W}, \quad (32)$$

$$R_R(W) = R_A(W)W = 1.$$

Short term and even fractional strategies are very risky since the Arrow–Pratt risk aversion index is very low. So we never allocate more than the f^* because then risk increases and growth decreases. If you allocate more than double the optimal fraction, then you will have a negative growth.

The mean variance theory has the deficiency that if the expectation $E_1 < E_2$ and variance $\sigma_1^2 < \sigma_2^2$, the theory cannot choose between the portfolios. Consider two-point probability distributions in Table 2, X_1, X_2 with mean and variance 1. They are indistinguishable by mean variance theory, but the capital growth

$$G(f_1^*) = \frac{1}{5} \ln \frac{1}{3} + \frac{4}{5} \ln 2 = 0.3348,$$

$$f_1^* = \frac{2}{3},$$

$$G(f_2^*) = \frac{1}{10} \ln \frac{1}{4} + \frac{9}{10} \ln \frac{3}{2} = 0.2263,$$

$$f_2^* = \frac{3}{8},$$
(33)

$G(f_2^*) < G(f_1^*)$, so we choose portfolio 1. Consider the growth function

$$G(f_1, f_2) = \frac{1}{50} \left[36 \ln \left(1 + \frac{3}{2}f_1 + \frac{4}{3}f_2 \right) + 4 \ln \left(1 + \frac{3}{2}f_1 - 2f_2 \right) \right. \\ \left. + 9 \ln \left(1 - f_1 + \frac{4}{3}f_2 \right) + \ln(1 - f_1 - 2f_2) \right]. \quad (34)$$

Table 3. Times of final wealth after 50 years with geometric mean return.

return	0.1	0.15	0.2	0.25	0.3
times	117.39	1.08e+003	9.10e+003	7.00e+004	4.97e+005

If we impose the constraints $f_1 + 2f_2 \leq 1, f_1 \geq 0, f_2 \geq 0$, X_1, X_2 are independent, so

$$\begin{aligned} E \sum f_i X_i &= f_1 + f_2, \\ \sigma^2 \sum f_i X_i &= f_1^2 + f_2^2. \end{aligned} \quad (35)$$

Let

$$L = f_1^2 + f_2^2 - \lambda(f_1 + 2f_2 - 1), \quad (36)$$

we have $f_1 = \frac{1}{5}, f_2 = \frac{2}{5}$, it has no definition for Equation (34). The mean variance theory only use probability information about first and second moments, however, the real world applications need more detailed distribution information. The criterion $E \ln(\cdot)$ use higher moment information, can provide a better solution.

Suppose outcomes of wealth relative random variable are R_1, \dots, R_n , with relevant probabilities p_1, \dots, p_n . The sample geometric mean return (SGMR) is defined from

$$\text{SGMR} = \prod_{i=1}^n (r_i)^{1/n} - 1 \quad (37)$$

The population geometric mean return (PGMR) can be written as

$$\text{PGMR} = \prod_{i=1}^n (r_i)^{p_i} - 1 \quad (38)$$

It is clear that, as $n \rightarrow \infty$, $\text{SGMR} \rightarrow \text{PGMR}$, *a.s.* When variable is discrete, there is a strategy named geometric mean strategy (GMS), which maximizes the PGMR. when the random variable is discrete, it is easy to show

$$\ln(1 + \text{PGMR}) = \sum_{i=1}^n p_i \ln r_i = E[\ln X]. \quad (39)$$

So, Log strategy and GMS are the same strategies, maximizing PGMR is equivalent to maximizing $E[\ln X]$. For any strategy

$$\text{PGMR} = e^{E[\ln X]} - 1. \quad (40)$$

The mean variance theory is based on arithmetic mean, while $G(\cdot)$ use geometric mean return. Hence, practical applications to long sequences of investments with multi-period and reinvestment are especially appropriate for the GMS. Hedge fund trading that enters and exits in a few seconds is such an application. Table 3 shows that small change of geometric mean return will lead to different performances. The final wealth of Buffett is 23,423 times the initial wealth after 40 years with geometric mean return 28.6%.

The main advantage of the criterion is the long run growth rate maximization, however, log is the most risky utility function, which provides a very high volatility of wealth level and it is most

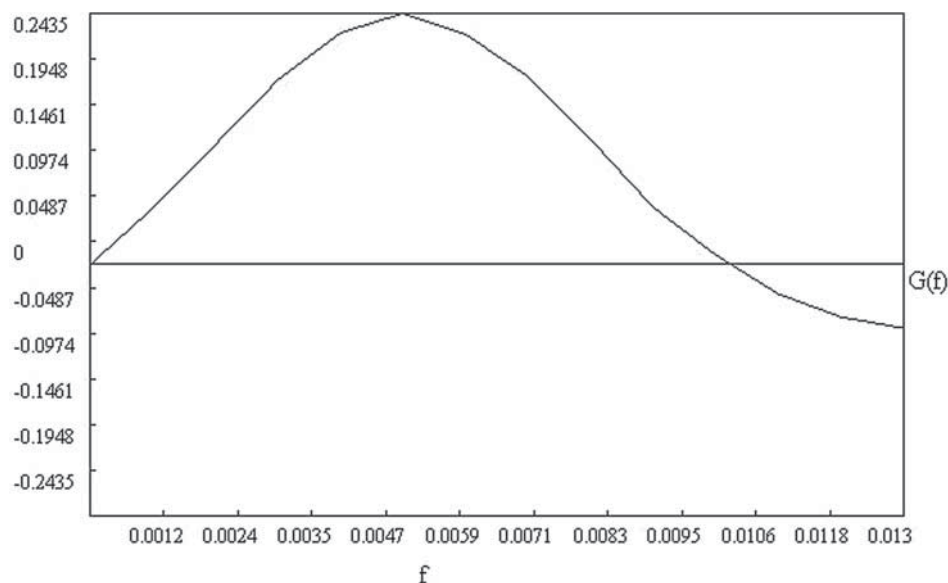


Figure 1. Relationship between growth and fraction.

dangerous. Hence, the criterion can be very risky in the short run. However, if used properly in situations where it is appropriate, it has wonderful properties. For long-term trader, the criterion yields the highest long run levels of wealth. The proof of optimal capital growth investment criterion is presented in Appendix 1.

4.2 The feasible allocation

The analytical solutions cannot be obtained, so we test our results on real financial data. We use the daily data extracted from our database, which is from US stock market covering the period 1 January 2000 – 31 December 2011. The stocks is obtained from these sectors: technology, finance, health care and consumer goods, media, automobile and energy, manufacture and retail, ETF. The pool portfolio was drawn at random from the assets pool, details see footnote¹.

We assume that there are no transaction cost, tax and short sell, which will be investigated in a separate work. To illustrate the relationship between capital growth and fraction, a trading model named Turtle [4] was applied into the markets. For simplicity, we assume each stock has the same allocation fraction, the investment optimization is simplified to a one-variable problem.

The nature of the objective function is now apparent and a graph of $G(f)$ versus f appears as shown in Figure 1. As can be seen from the figure, $G(f)$ has a unique maximum at $f = 0.0055$. The feasible fraction is located in $[0, 0.011]$, we will have a negative growth when allocation fraction more than 0.011.

5. Conclusions

We investigate the problem of dynamic optimal capital growth of diversified investment. A general framework that one strives to maximize the expected log utility of long-term growth rate was developed.

The results show that the investor's wealth will exceed the initial value when the fraction is chosen less than critical value. But, if larger than the value, ruin is almost sure. In order to

maximize wealth, we should choose the optimal fraction at each trade. Empirical results with real financial data show the feasible allocation set, which means that capital growth portfolio does not risk ruin, either in short run or long run. If we want more higher growth rate, the greater the fraction will be accepted.

Future work could address the extension of the model to consider draw down and transaction costs constraints.

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Note

1. The pool portfolio include 50 assets from USA market: CSCO, AMZN, AAPL, YHOO, MSFT, C, ADS, BRK.A, BAC, AFL, KO, SBUX, JNJ, KFT, PFE, CBS, TWX, DIS, TRI, AOL, DUK, RIO, XOM, AEP, COP, CL, GE, PG, AA, DD, DIA, XLE, XLV, USO, FXI, CPHI, SNP, BIDU, PTR, CHL, AES, ORCL, QCOM, EBAY, GOOG, GS, MA, WMT, FDX, RTN

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Appendix. Proof of optimal capital growth investment criterion

THEOREM A.1 (a) If $G(f) > 0$, then $\lim_{n \rightarrow \infty} W_n = \infty$ almost surely, for each wealth target M , $\Pr[\liminf_{n \rightarrow \infty} W_n > M] = 1$. (b) If $G(f) < 0$, then $\lim_{n \rightarrow \infty} W_n = 0$ almost surely, for each small number ϵ , $\Pr[\limsup_{n \rightarrow \infty} W_n < \epsilon] = 1$.

Proof (a) By the Borel strong law, $\lim_{n \rightarrow \infty} (1/n) \ln(W_n/W_0) = G(f)$, and $G(f) > 0$, $\exists N \in \mathbb{Z}^+$, $\forall n > N$,

$$\frac{1}{n} \ln \frac{W_n}{W_0} \geq \frac{G(f)}{2} \quad (\text{A1})$$

so

$$W_n \geq W_0 e^{nG(f)/2} \quad (\text{A2})$$

$\forall n > N$, so, $\lim_{n \rightarrow \infty} W_n = \infty$. (b) It is similar to (a), $\lim_{n \rightarrow \infty} W_n = 0$. ■

The trader's wealth will exceed the initial value when f is chosen in the interval $(0, f_c)$. But, if $f > f_c$, ruin is almost sure. In order to maximize wealth, we should maximize $E[\log(1 + f_i r_i)]$ by choosing the optimal fraction f^* at each trial although the probabilities change from one trial to the next.

THEOREM A.2 *The capital growth trader never risks ruin.*

Proof Denoting the average compound return over T periods by $e^{G(f_1, f_2, \dots, f_n)} - 1$, the probability of which less than zero was defined as ruin risk of capital growth. According to Kelly properties,

$$G(f_1, f_2, \dots, f_n) > 0 \quad f_i \in (0, f_i^*) \quad (\text{A3})$$

so

$$\text{Prob}\{[e^{G(f_1, f_2, \dots, f_n)} - 1] \leq 0\} = 0 \quad f_i \in (0, f_i^*). \quad (\text{A4})$$

This means that capital growth portfolio does not risk ruin, either in short run or long run. ■